A Flexible, Scalable and Provably Tight Relaxation for Matching Problems

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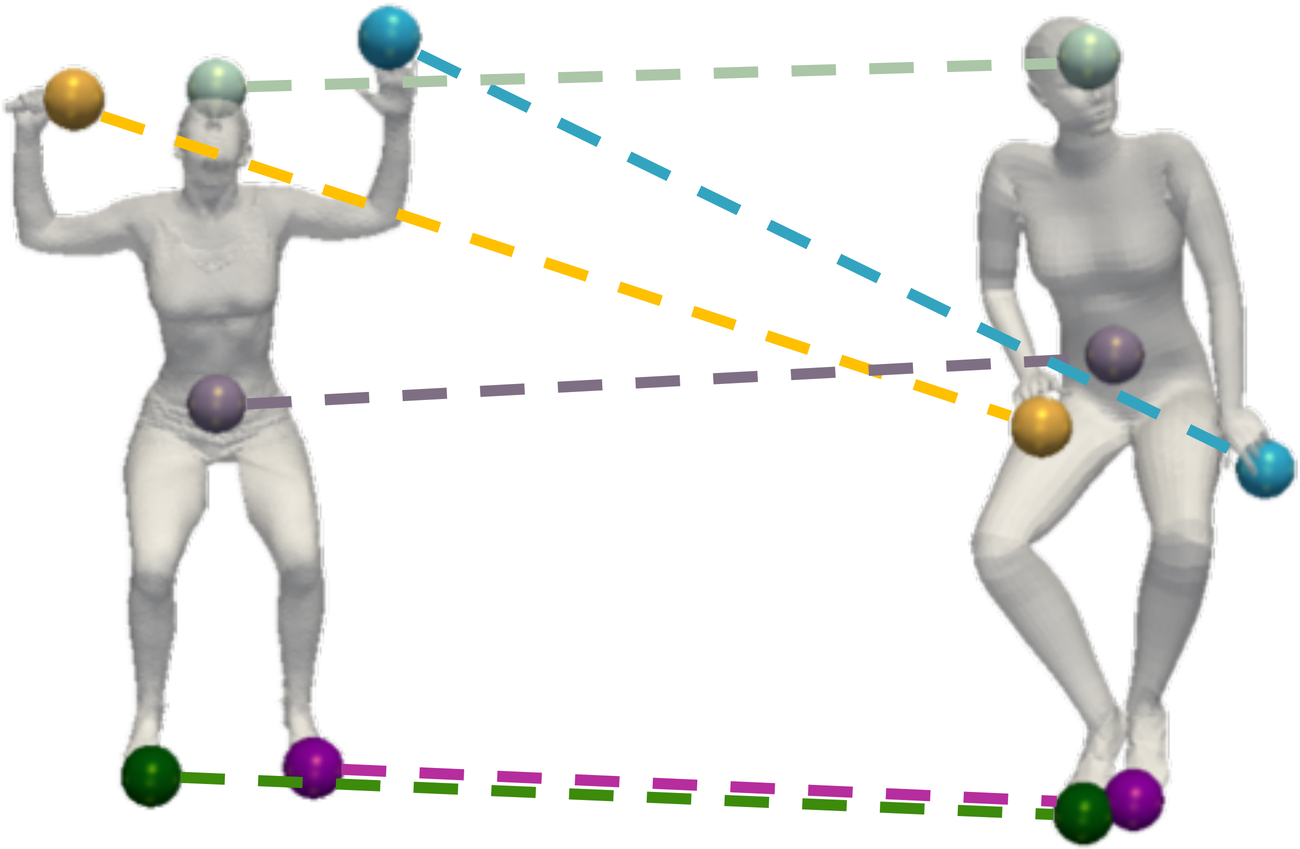
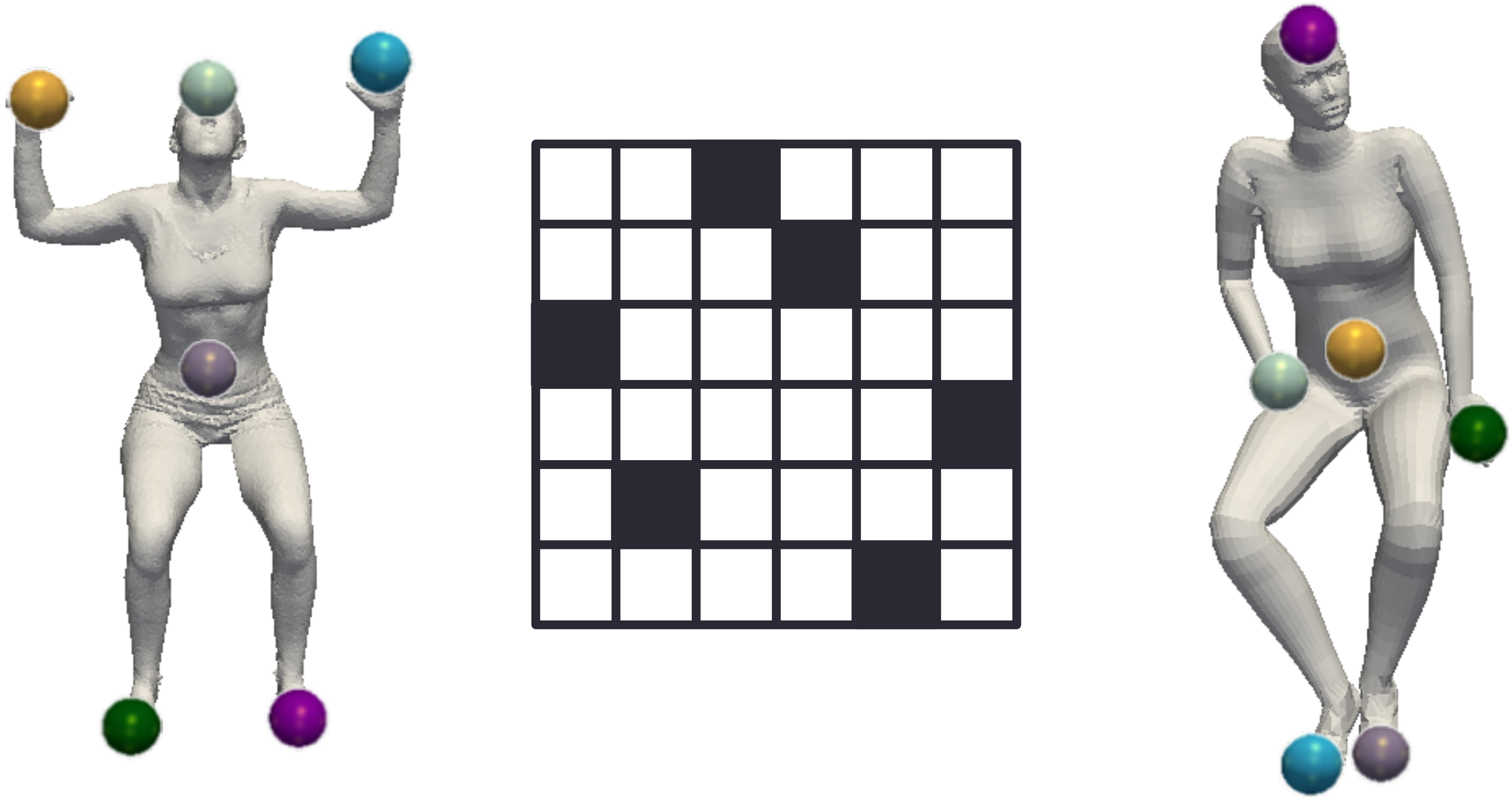
Matching

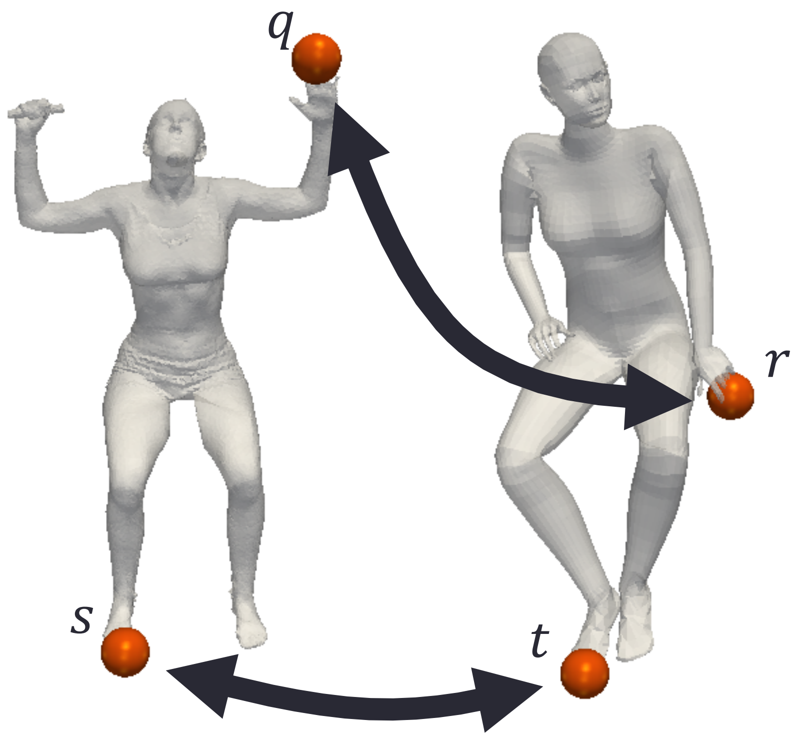


Representing Correspondences



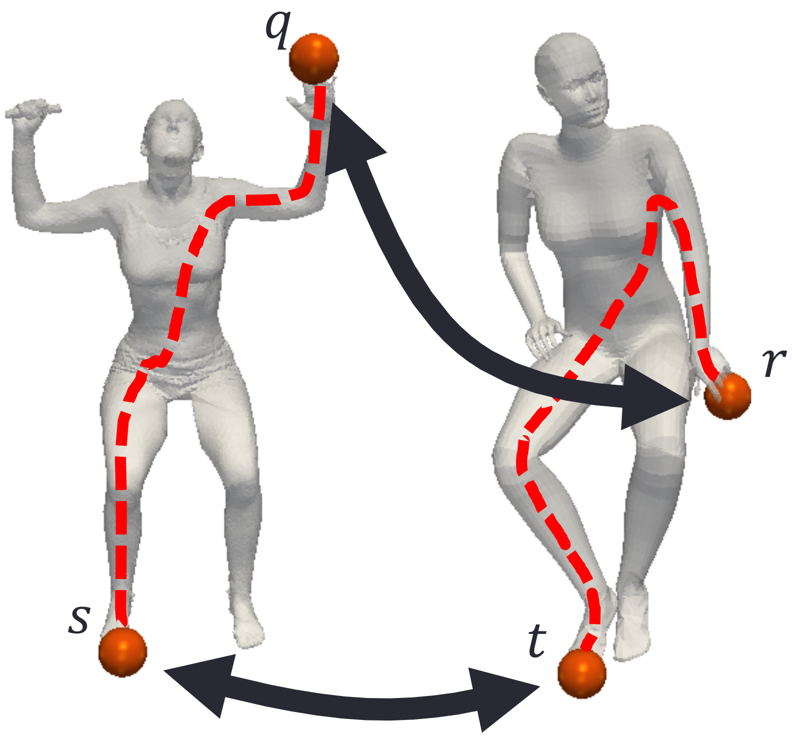
Quadratic Matching

penalty for a pair of matches = $W_{qr,st}$



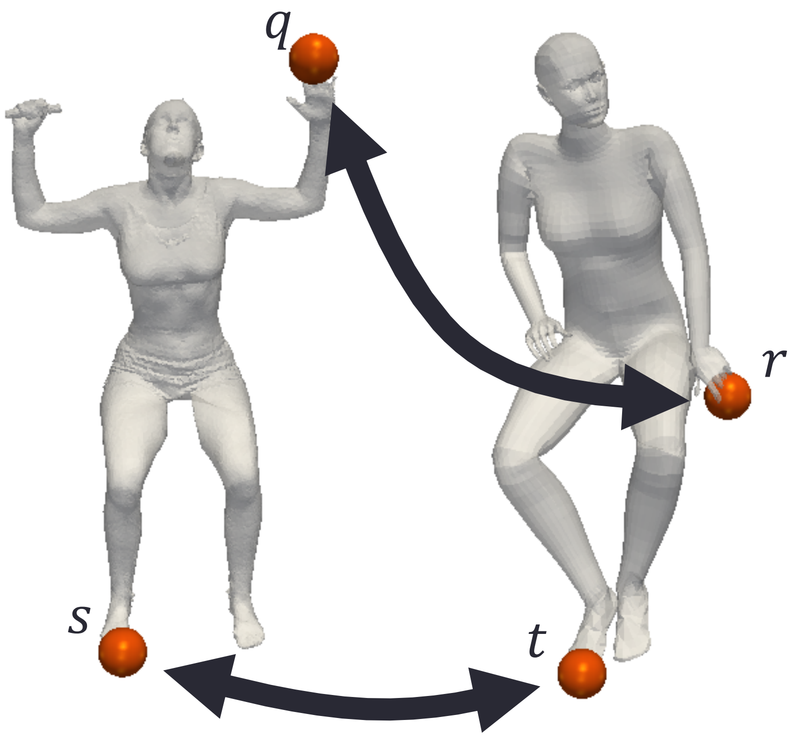
Quadratic Matching

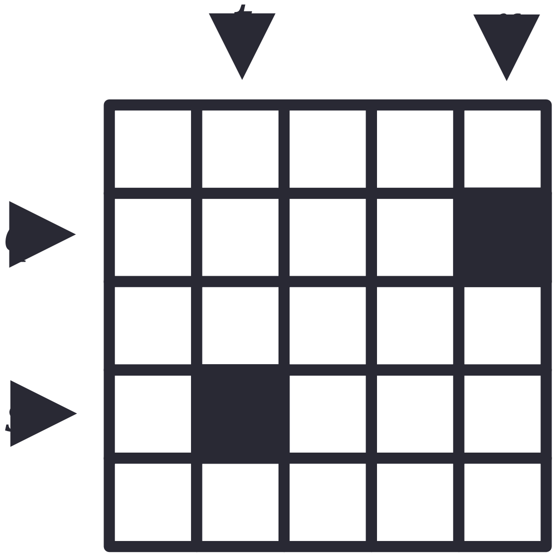
$$W_{qr,st} = |d_{qs} - d_{rt}|$$



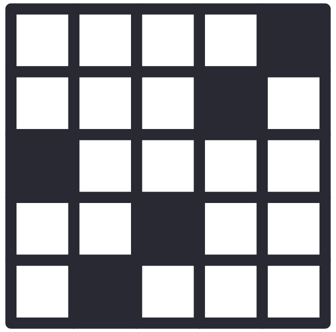
Quadratic Matching

$$\min_{X\in\Pi}\sum_{q,r,s,t} W_{qr,st} X_{qr} X_{st}$$





Quadratic Matching



X

$\Leftrightarrow$

$[X]$

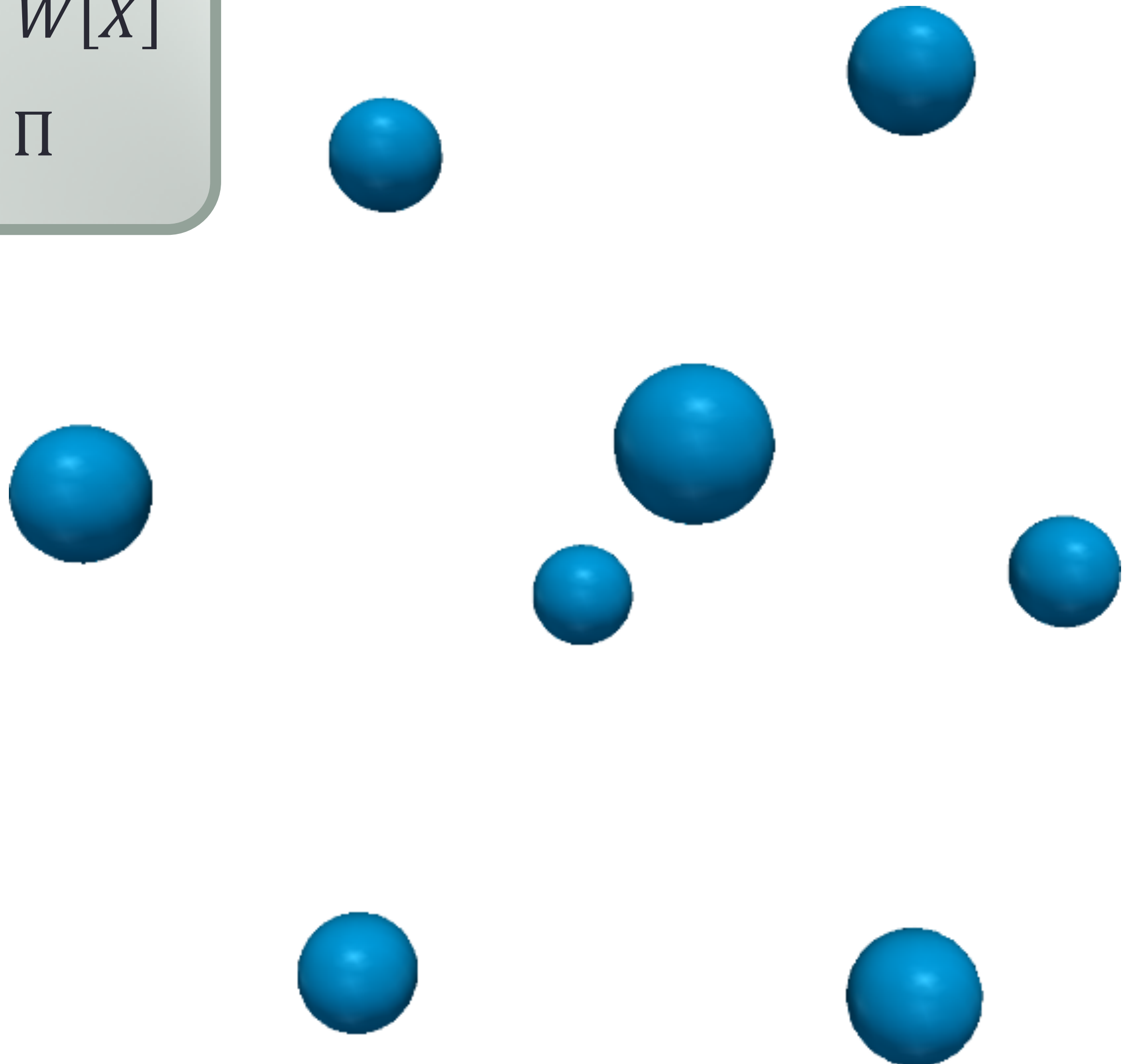
$$\min_{X \in \Pi} \sum_{q,r,s,t} W_{qr,st} X_{qr} X_{st}$$

$$\min_{X \in \Pi} [X]^T W [X]$$

The Challenge

$$\begin{array}{ll} \min\limits_{X} & [X]^T W [X] \\ \text{subject to} & X \in \Pi \end{array}$$

- Non-convex objective
- Non-convex domain
- NP-hard problem

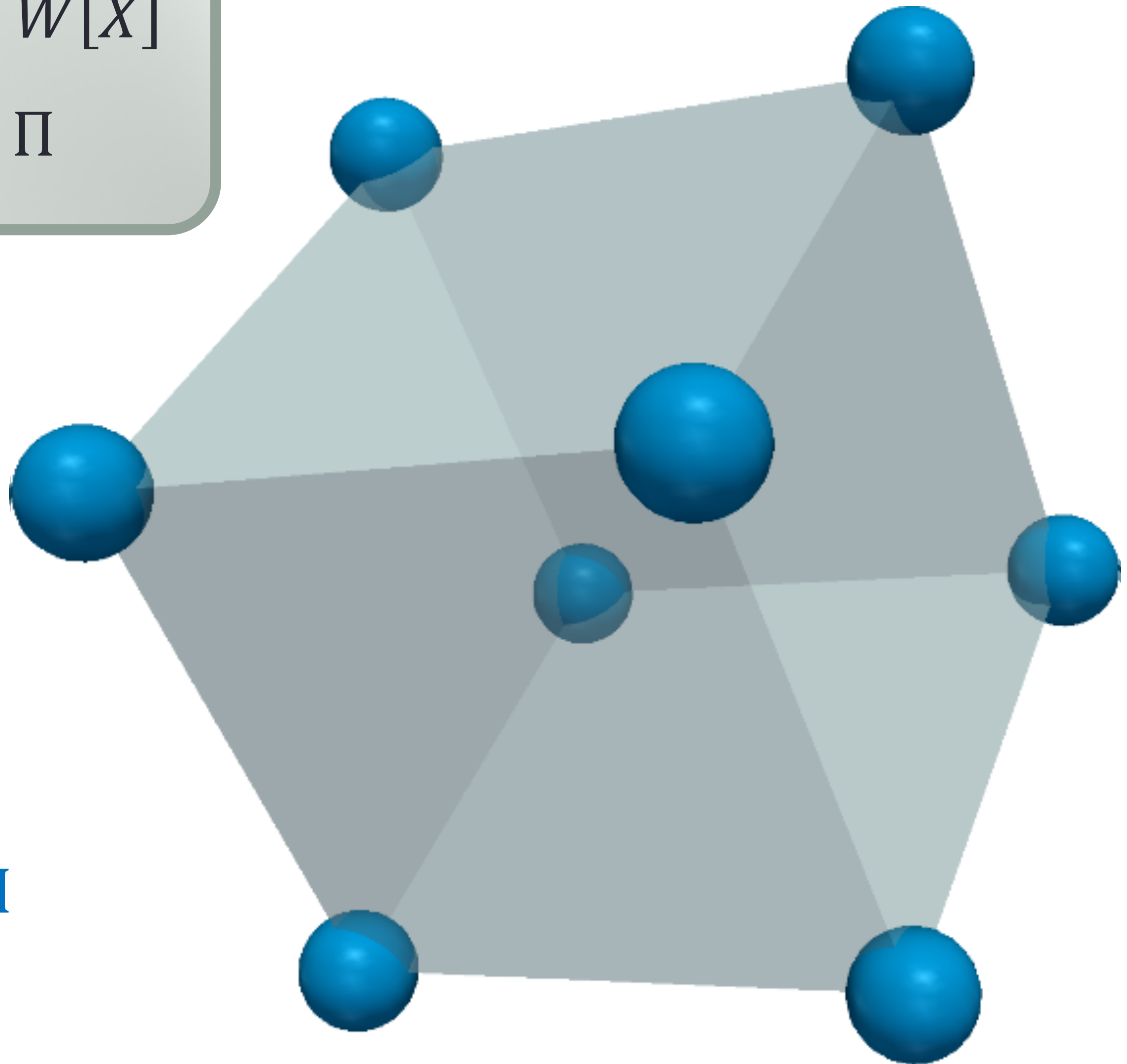


Doubly Stochastic Relaxation

$$\begin{array}{ll} \min\limits_{X} & [X]^T W [X] \\ \text{subject to} & X \in \Pi \end{array}$$

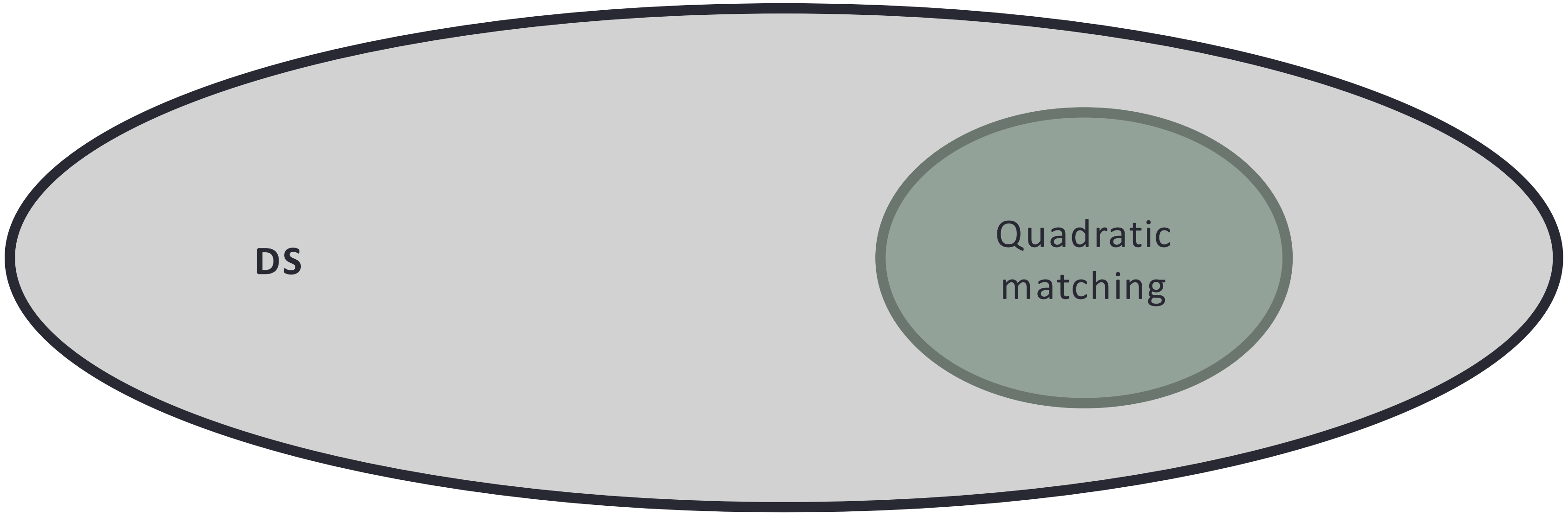
- Tractable for $W \succcurlyeq 0$

$X \in \text{conv}\Pi$



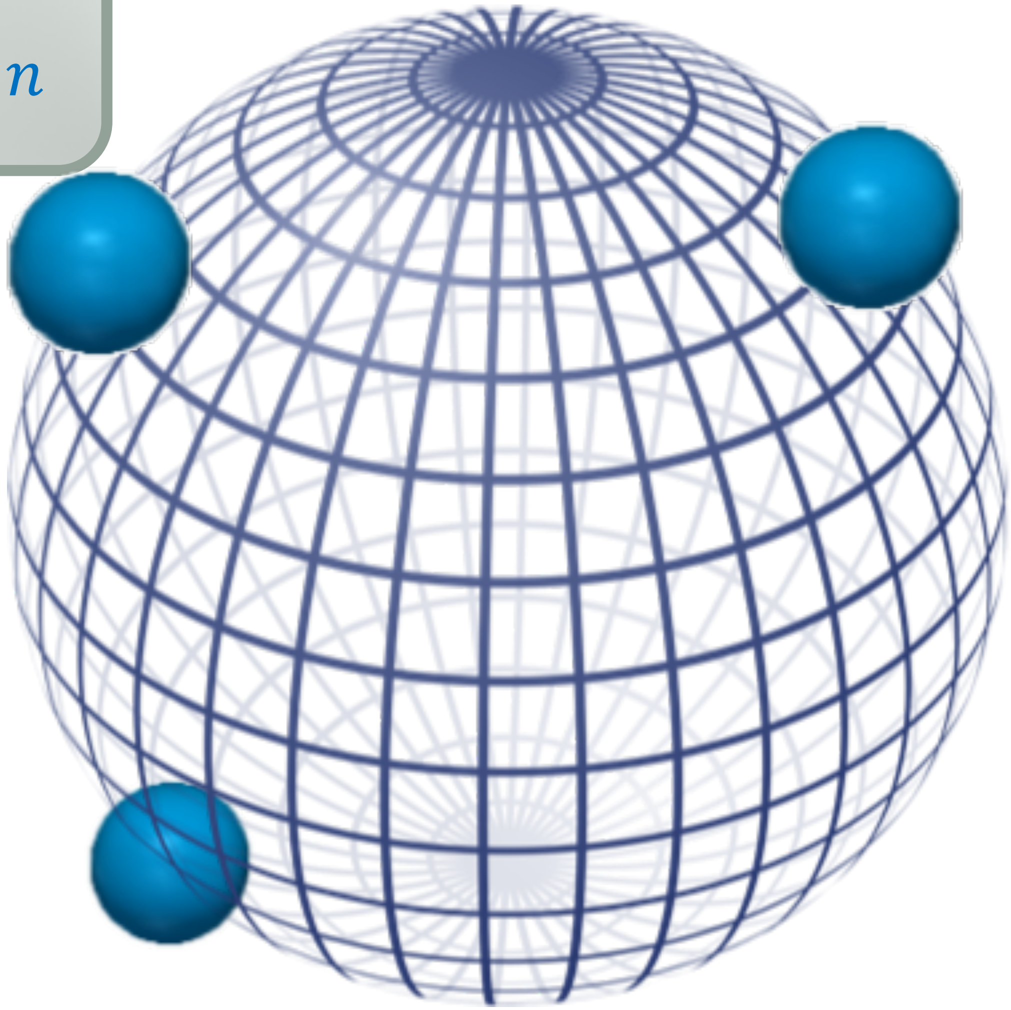
Doubly Stochastic Relaxation

$$\min_{X} \quad [X]^T W [X]$$
$$\text{subject to} \quad X \in \operatorname{conv}\Pi$$



Spectral Relaxation

$$\min_{X} \quad [X]^T W [X]$$
$$\text{subject to} \quad \|X\|_F^2 = n$$

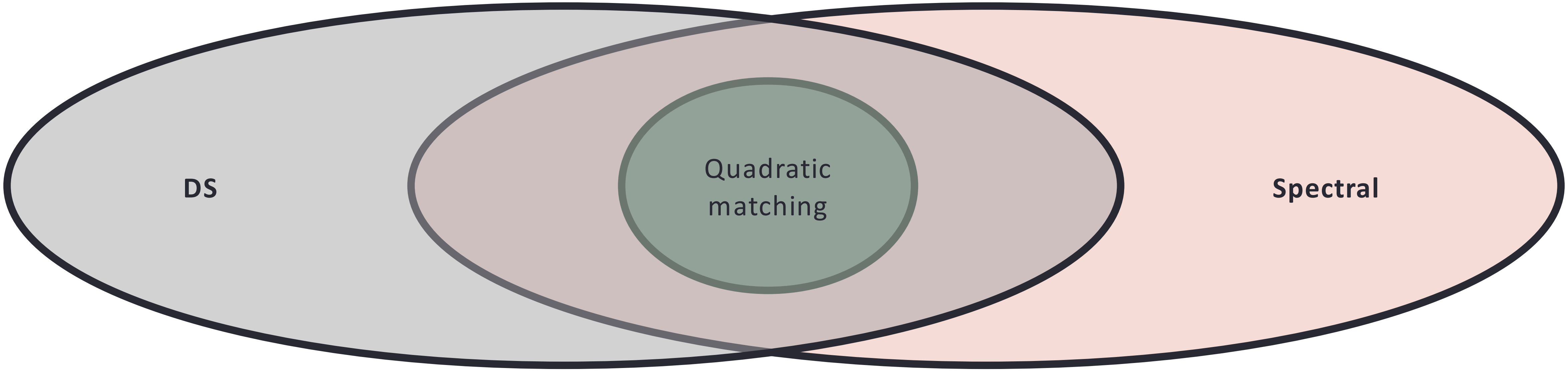


- Eigenvector problem

[Leordeanu & Hebert 2005]

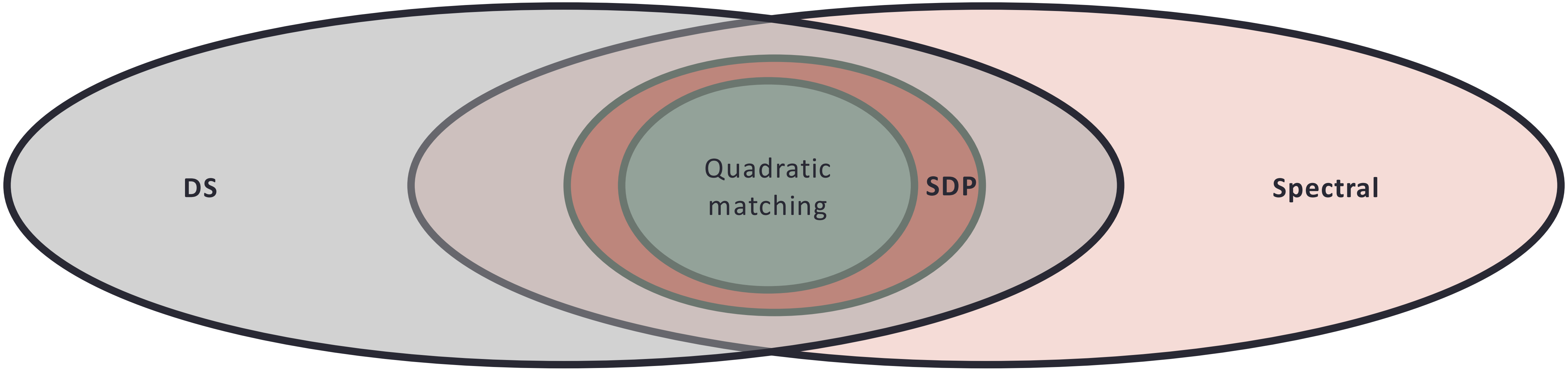
Spectral Relaxation

$$\begin{aligned} \min_{X} \quad & [X]^T W [X] \\ \text{subject to} \quad & \|X\|_F^2 = n \end{aligned}$$



SDP Relaxation

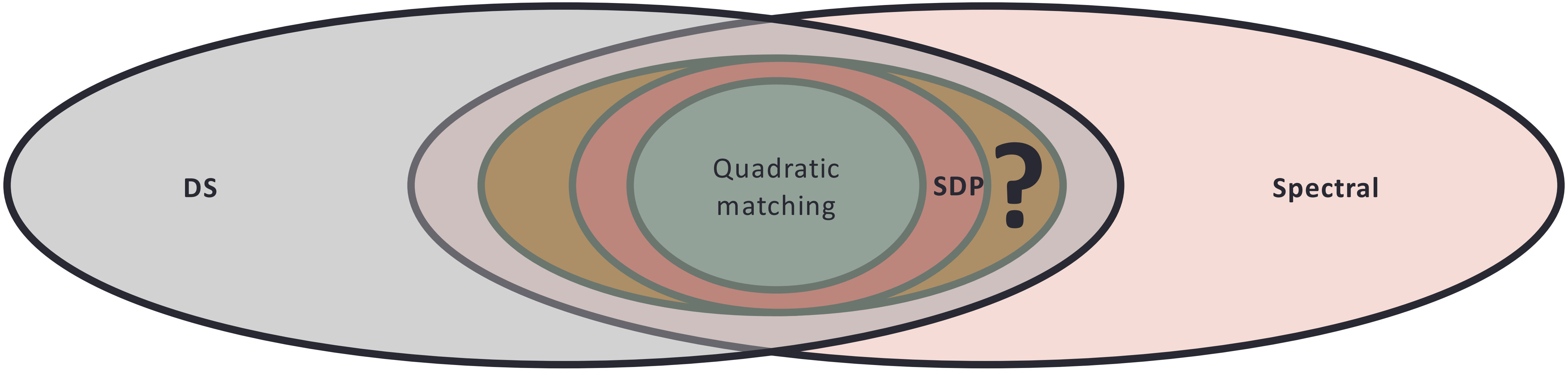
- Tight!
- Not scalable - $O(n^4)$ variables



[Zhao et al. 1998, Kezurer et al. 2015]

Question:

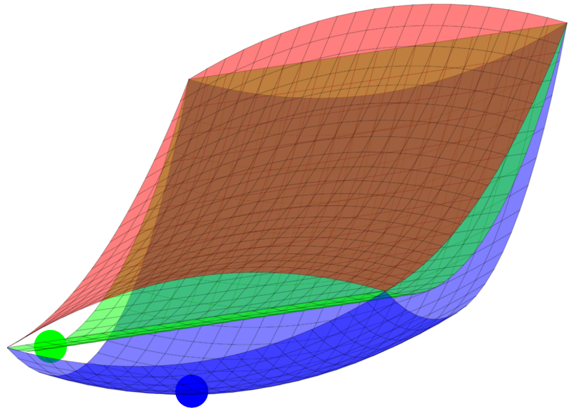
Can we find a tight relaxation without compromising scalability?



[Kezurer et al. 2015]

Our approach

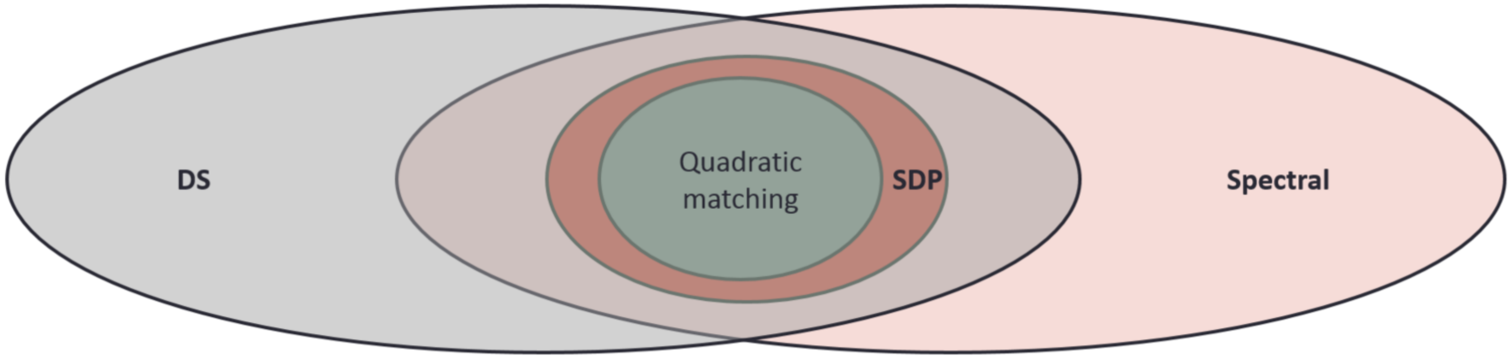
- Construct a parametric family of equivalent problems



- Choose optimal parameter value for relaxation

$$a = \lambda_{min}(W|_{aff(\Pi_n)})$$

- Place in relaxation hierarchy



Equivalent formulations

$$[X]^T W [X] - \underbrace{a(\|X\|_F^2 - n)}_{0}$$

$$X \in \Pi_\mathrm{n}$$

Relaxation

$$\underbrace{[X]^T W [X] - a\left(\|X\|_F^2 - n\right)}_{E(X,a)}$$

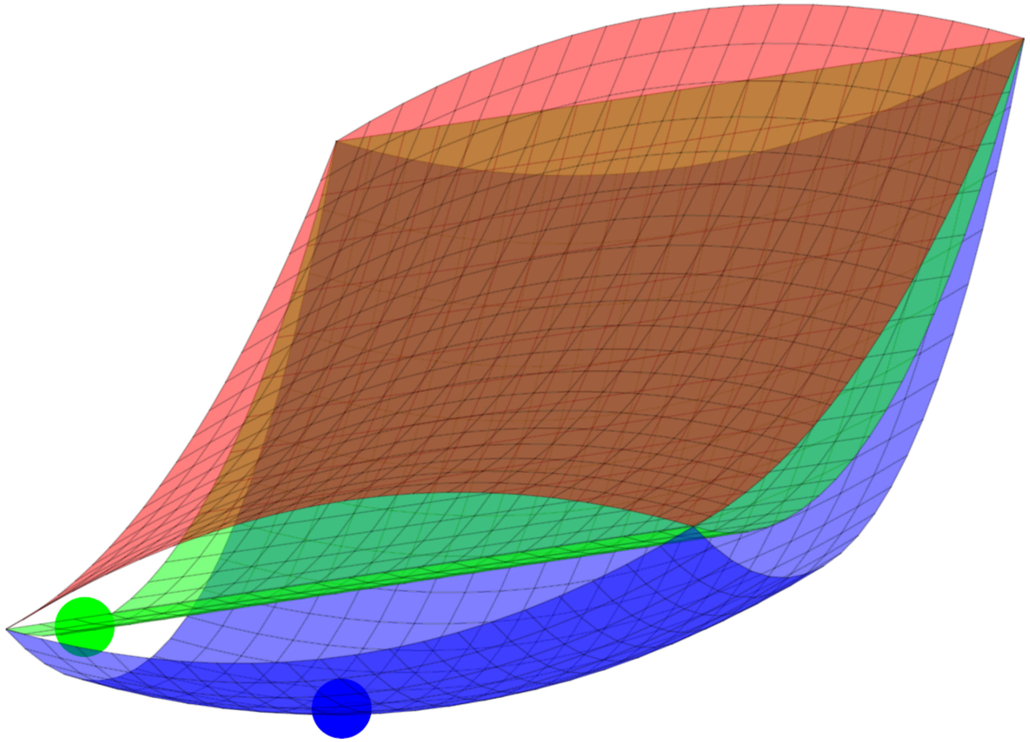
$$X \in \mathrm{conv}(\Pi_\mathrm{n})$$

Goal: Find convex relaxation that generates maximal lower bound

Optimal parameter value

Lemma

For $X \in conv(\Pi_n)$, $b > a$ we have $E(X, b) > E(X, a)$



$\Rightarrow$Take maximal a s.t. problem is convex

Optimal parameter value

Solution (1)

Take $a = \lambda_{min}(W)$

$\Rightarrow$Hessian is $W - \lambda_{min} I \Rightarrow$ convex

Can we do better?

[Fogel et al. 2013, 2015]

Optimal parameter value

Solution (2)

Take $a = \overline{\lambda_{min}} = \lambda_{min}\left(W|_{aff(\Pi_n)}\right)$

$$\Rightarrow \text{convex on } aff(\Pi_n)$$

Recap

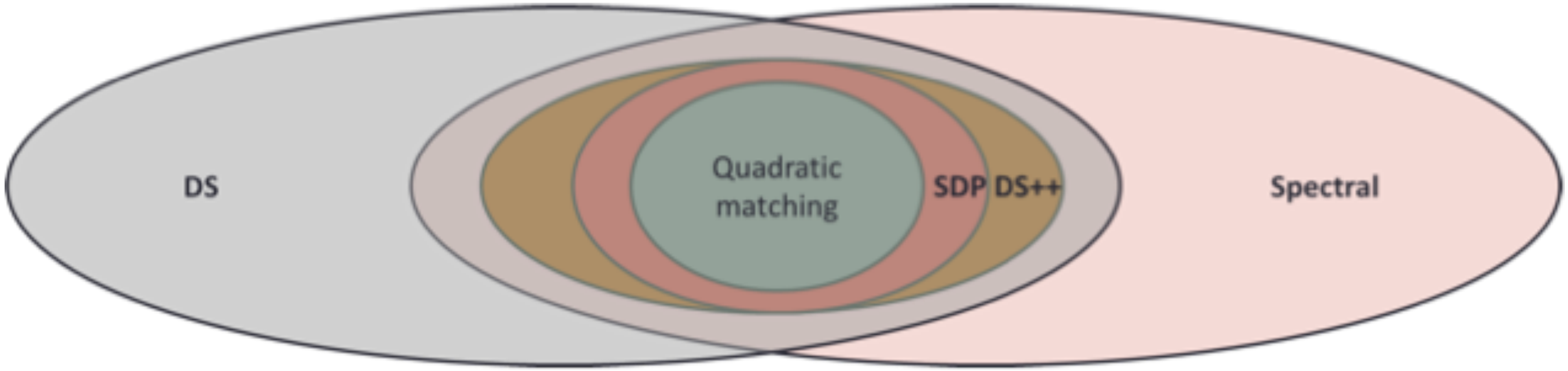
- We have found an optimal relaxation in the family we proposed. We call it **DS++**:

$$\begin{array}{ll} \min\limits_{X} & [X]^T W [X] - \overline{\lambda_{min}}\left(\|X\|_F^2 - n\right) \\ \text{subject to} & x \in conv(\Pi_n) \end{array}$$

- Is it a good relaxation?
  - We show it is
  - Method: compare all relaxations by “embedding” them in a high dim space

Relaxation hierarchy

- Establish a partial order on relaxations
- In our case: partial order == relaxation domain inclusion

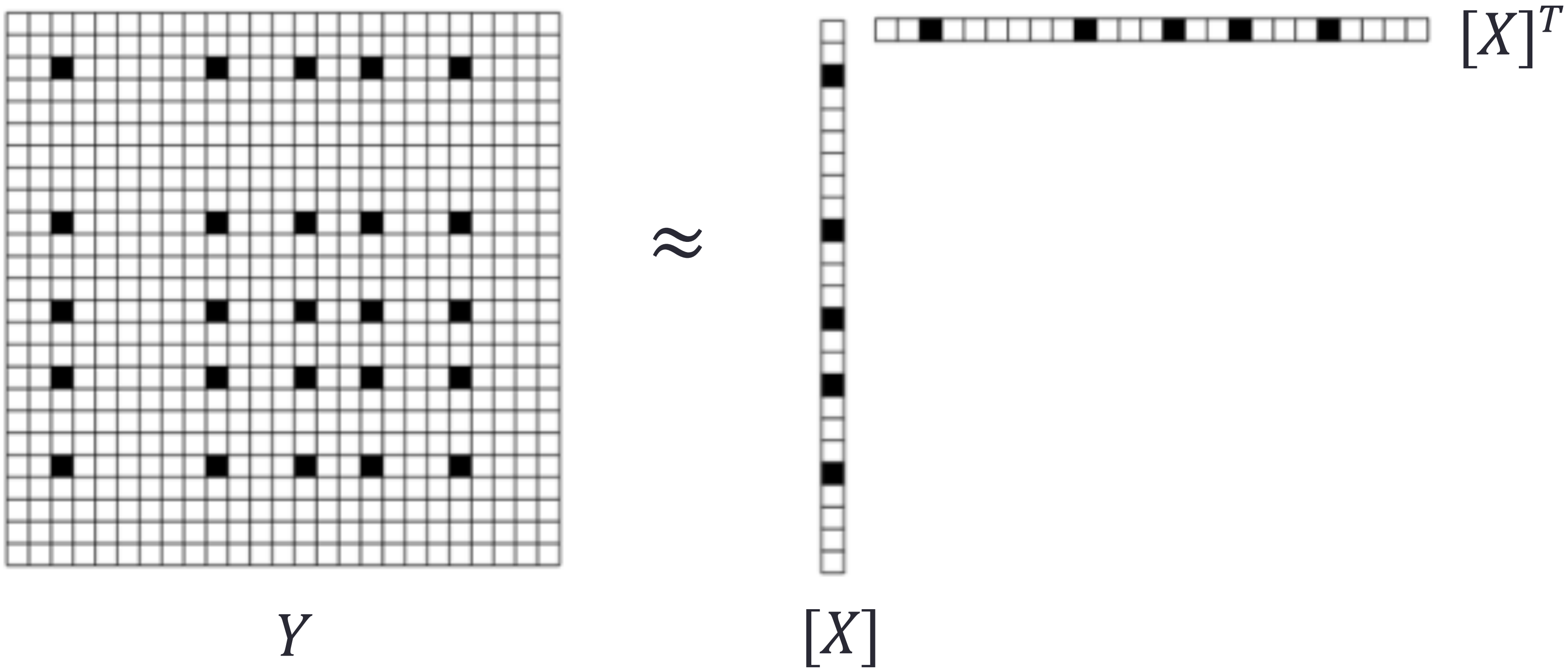


- Need to move to a common domain!

[Kezurer et al. 2015]

Relaxation hierarchy

- New variable (X, Y)
- Y represents quadratic monomials in X



Relaxation hierarchy

- The **doubly stochastic relaxation** as SDP:

$$\begin{aligned} \min_{X,Y} \quad & \operatorname{tr}(WY) \\ \text{subject to} \quad & Y \succcurlyeq [X][X]^T \\ & \mathrm{A[X]} = \mathrm{b}, \\ & [X] \geq 0 \end{aligned}$$

Similar to $\operatorname{tr}(WY) = \operatorname{tr}(W[X][X]^T) = [X]^T W [X]$

SDP constraint

Equivalent to $\mathrm{X} \in conv\Pi_n$

Relaxation hierarchy

- The **spectral relaxation** as SDP:

$$
\begin{array}{ll}
\min_{X,Y} & \mathrm{tr}(WY) \\
\text{subject to} & Y \succcurlyeq [X][X]^T \\
 & \mathrm{tr}Y = n
\end{array}
$$

Relaxation hierarchy

Theorem

DS++ is equivalent to the following:

$$\min_{X,Y} \quad \mathrm{tr}(WY)$$

$$\text{subject to} \quad Y \succcurlyeq [X][X]^T$$

$$\mathrm{Ax} = \mathrm{b}$$

$$[X] \geq 0$$

$$trY = n$$

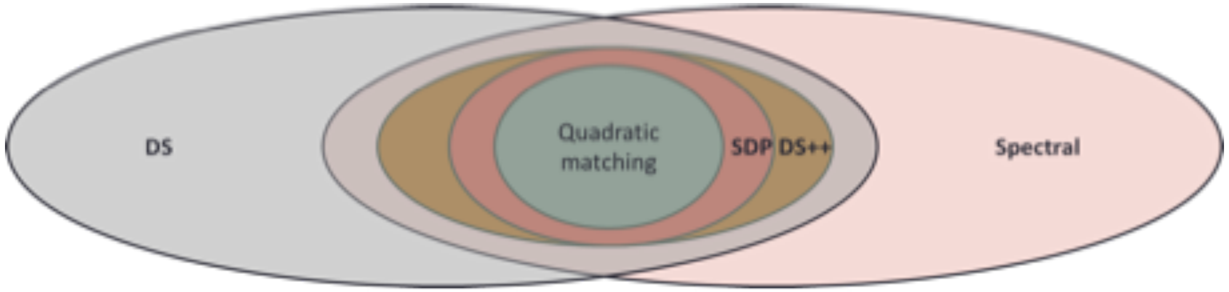
$$\mathrm{AY} = \mathrm{bx}^{\mathrm{T}}$$

***doubly** stochastic constraint*

***Spectral** constraint*

***Additional n^3 constraints**!*

Relaxation hierarchy



Corollary (1)

DS++ is more accurate than both the **DS** and **Spectral** relaxations!

Corollary (2)

DS++ is less accurate than [Kezurer 15']

SDP relaxation in n^4 variables

$$\min_{X,Y} \quad \mathrm{tr}(WY)$$

subject to
$$Y \succcurlyeq [X][X]^T$$
$$\mathrm{Ax} = \mathrm{b}$$
$$[X] \geq 0$$
$$trY = n$$
$$\mathrm{AY} = \mathrm{bx}^{\mathrm{T}}$$

quadratic program in n^2 variables

$$\min_{X} \quad [X]^T W[X] - \lambda_{min}(\|X\|_F^2 - n)$$

subject to
$$\mathrm{x} \in conv(\Pi_n)$$

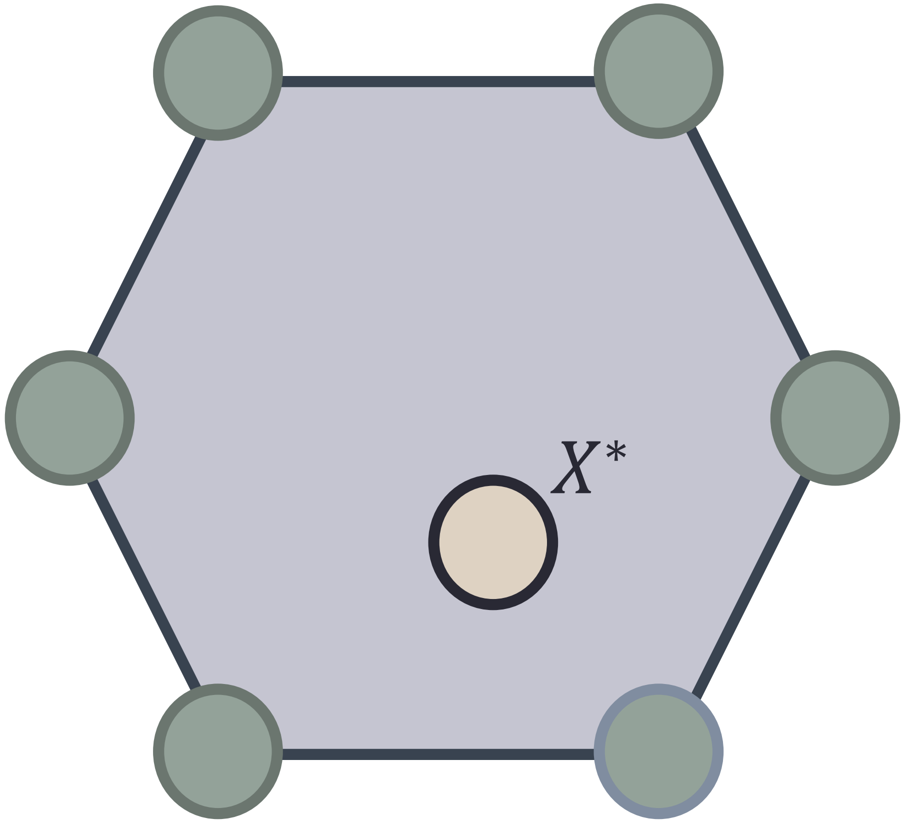
Tight!

Fast!

Projection

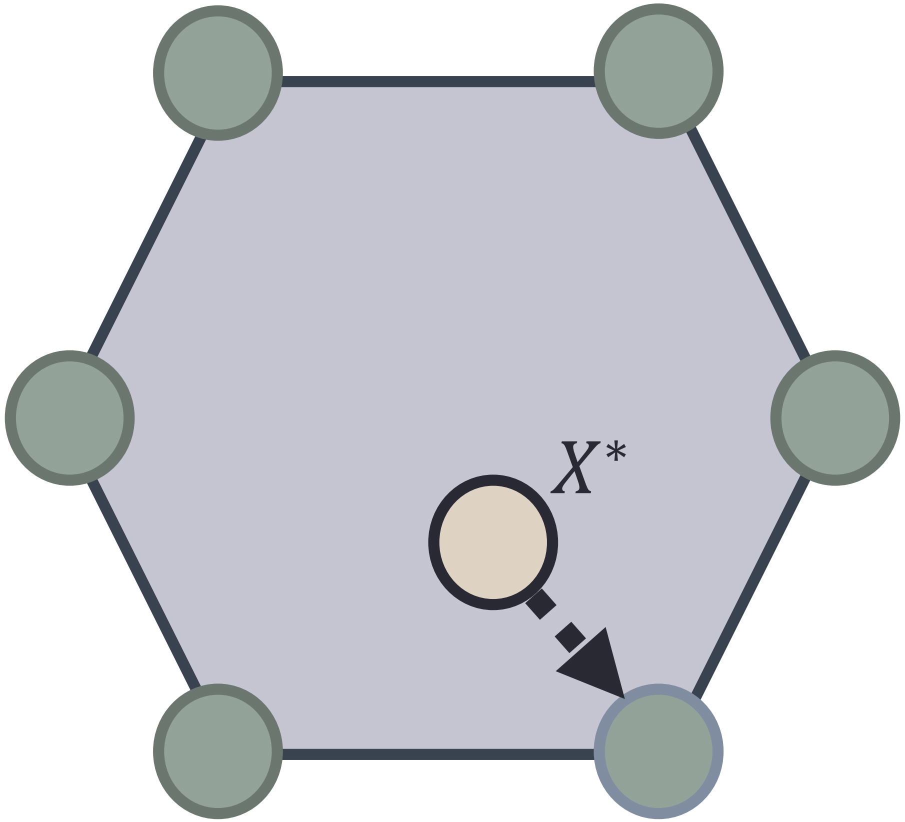
Natural projection

- **Problem**: what if X^* is not a permutation matrix?



Natural projection

- **Problem:** what if X^* is not a permutation matrix?
- **Common solution:** L_2 projection – does not take functional into account

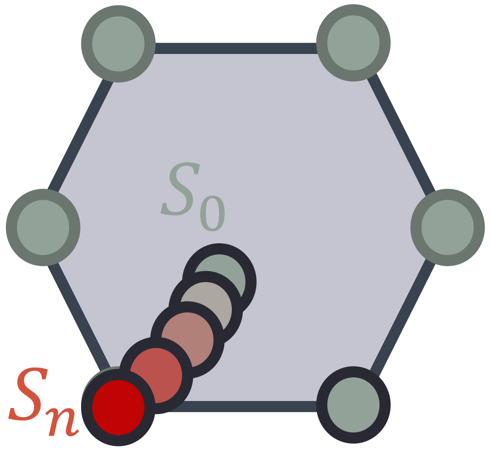


Natural projection

- Our solution :
  - Solve convex relaxation $E(X, a)$ for optimal $a_0 = \lambda_{min}$.
  - gradually deform objective from convex to concave by increasing a

convex ... concave

$E(X, a_0)$ ... $E(X, a_n)$



  - Concave objective – guaranteed to get a permutation!
  - We use [Solomon et al. 2016] for optimization

[Ogier and Beyer 1990; Zaslavskiy et al. 2009].

Natural projection

convex

concave

$E(X, a_0)$ ... $E(X, a_n)$

Applications

Applications: shape matching

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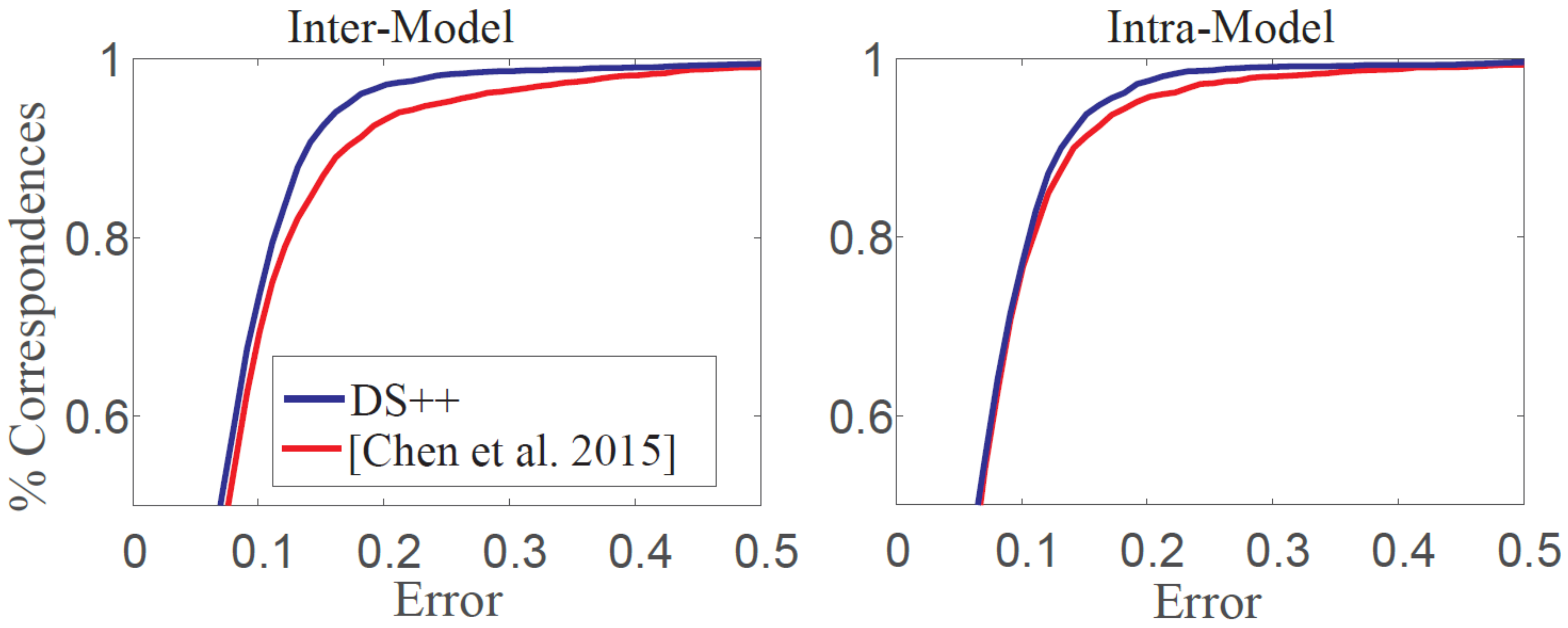
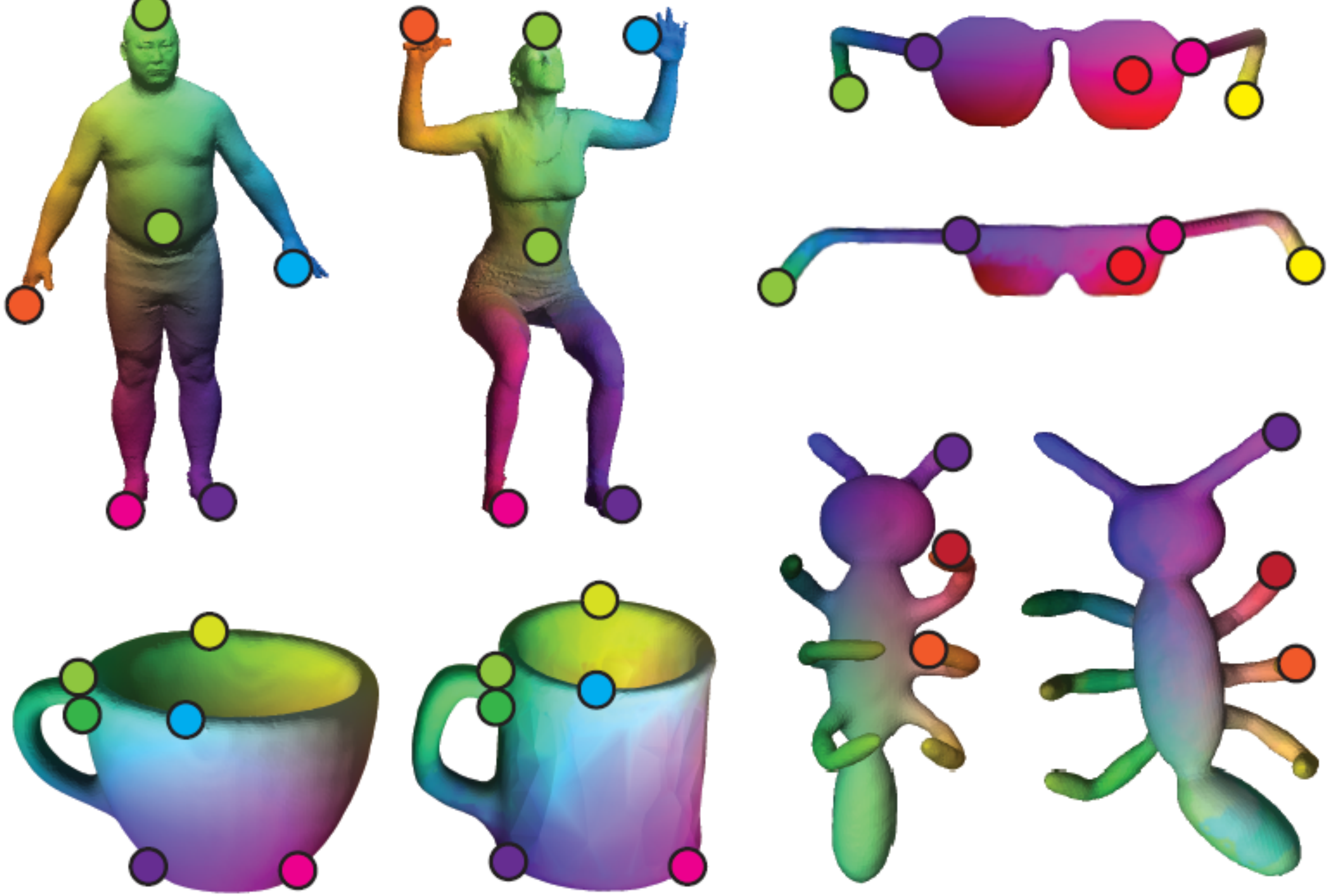
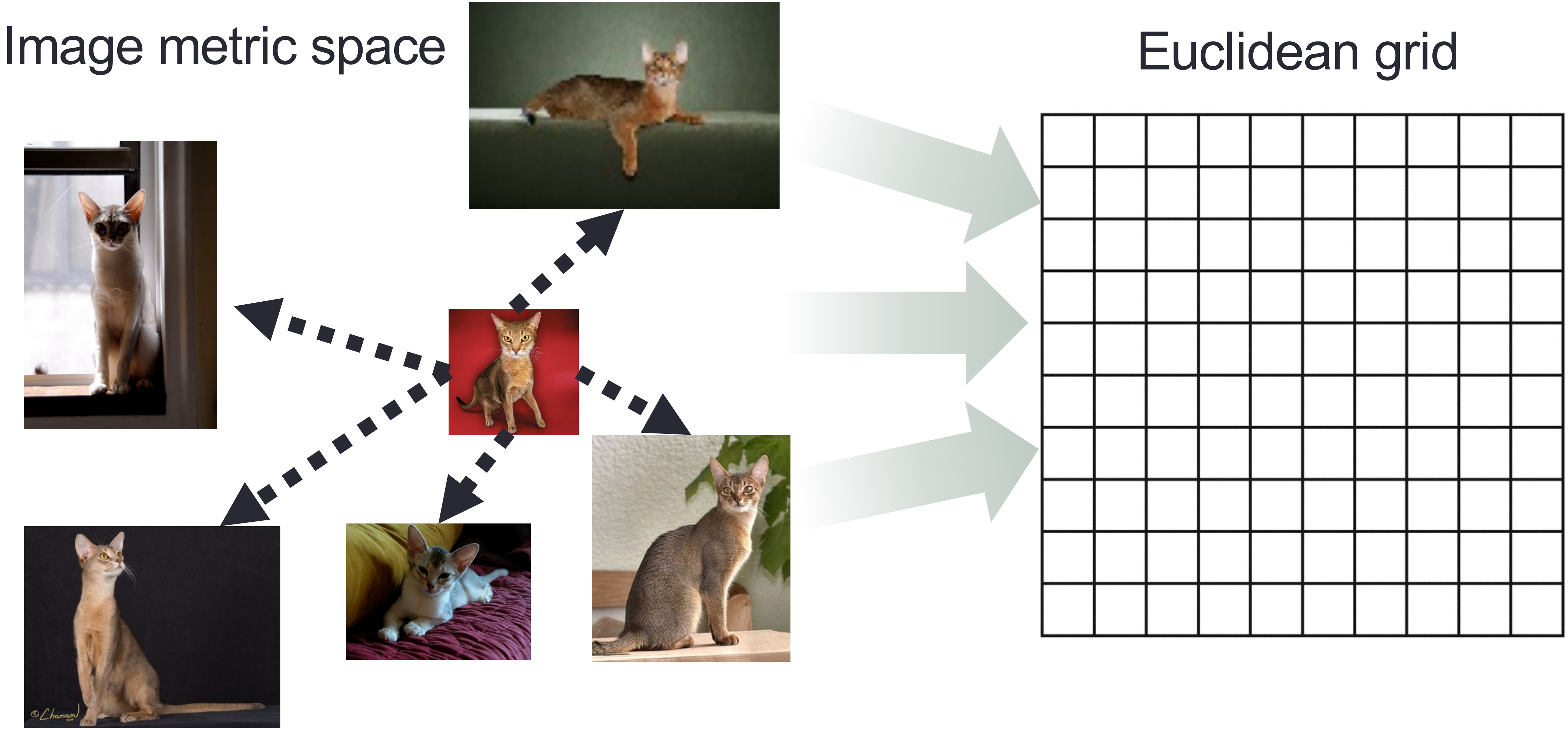


Applications : shape matching



Applications : image arrangement



Applications : image ordering

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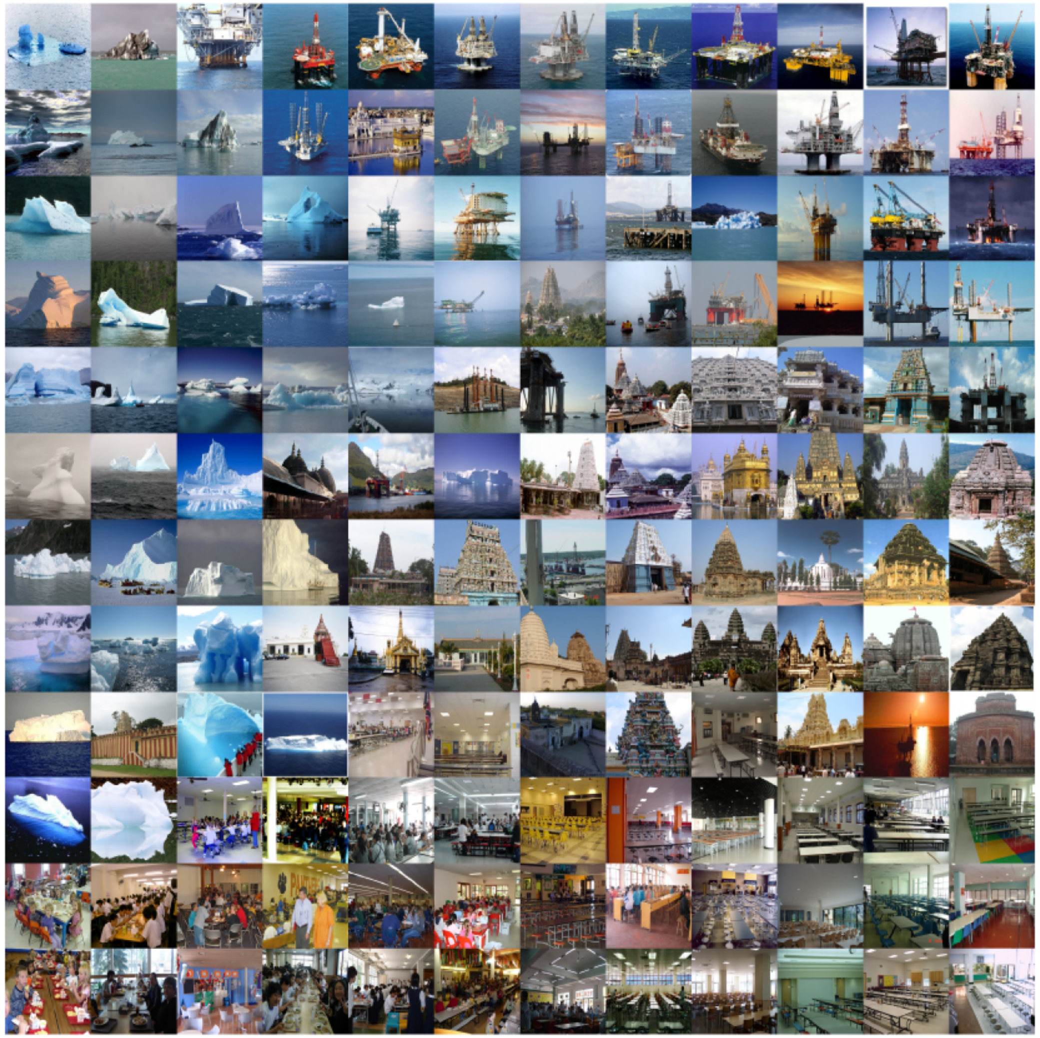


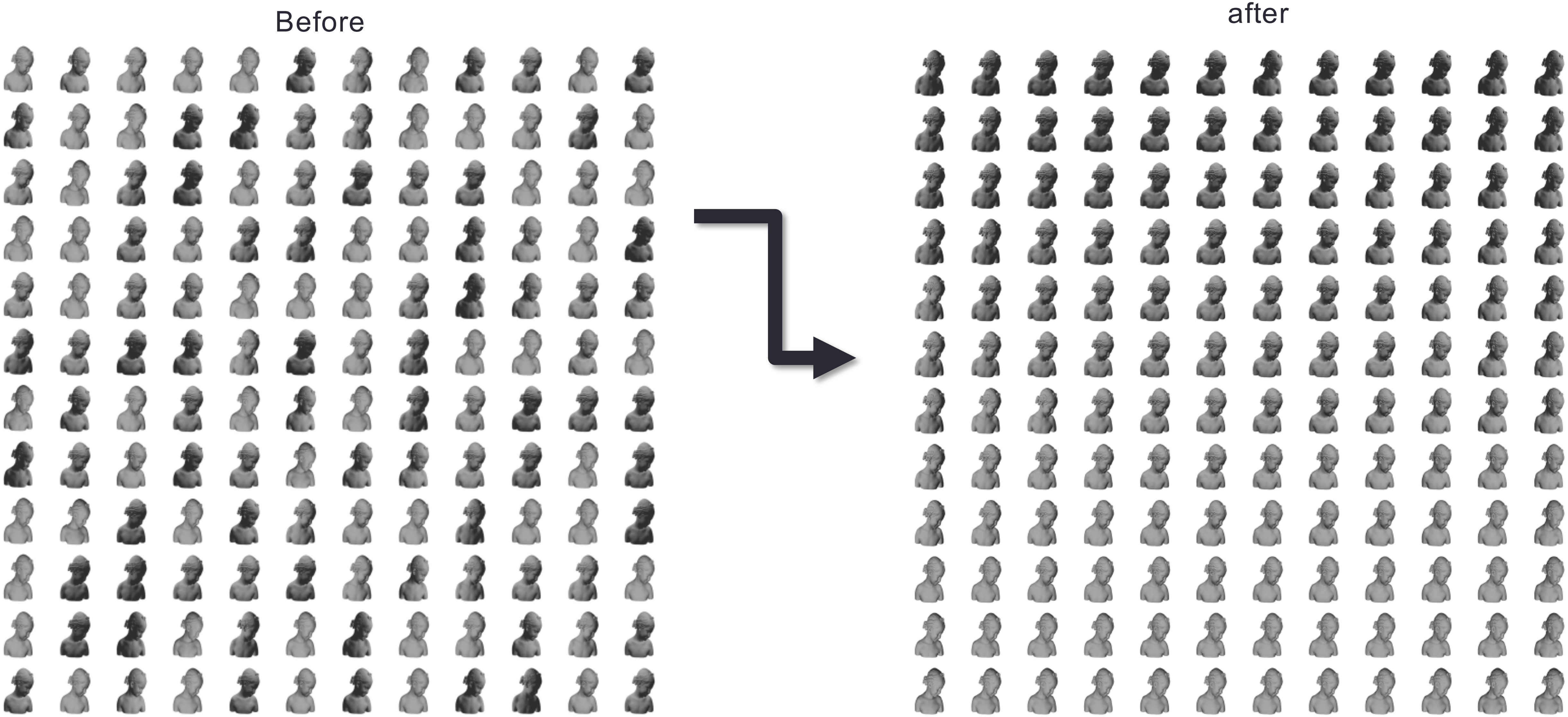
Icebergs

Classrooms

Oil rafts

Temples

Applications : image ordering



Conclusion

- More accurate relaxation at the same complexity
- Natural projection method
- Works on all convex and concave energies

Limitations / future work

- Best relaxation in n^2 variables?
- Partial matching
- Optimization with Frank-Wolfe scheme

The End

- Code is available online:

http://www.wisdom.weizmann.ac.il/~haggaim/

- Support
  - ERC Starting Grant (SurfComp)
  - Israel Science Foundation
  - I-CORE

- Thanks for listening!